Nomination form for the 2021 Nishina Asia Award

Candidate (name, affiliation, curriculum vitae including the date of the degree of Ph.D., nationality, address, email and telephone)
Song He, Professor, Institute of Theoretical Physics, Chinese Academy of Sciences Nationality: China Address: 55 Zhongguancun East St. Haidian District, Beijing, 100190, China Email: songhe@itp.ac.cn Tel: 010-62552084 Homepage: itp.ac.cn googlescholar Academic Positions: 2015- present Faculty at Institute of Theoretical Physics, Chinese Academy of Sciences 2012-2015 Postdoc fellow at Perimeter Institute (Waterloo, Canada) and Member at Institute for Advanced Study (Princeton, USA) 2009-2012 Postdoc fellow at Albert-Einstein Institute (Potsdam, Germany) Education: 2005-2009 School of Physics, Peking University, PhD degree on Theoretical Physics received on 1sst July 2009 2002-2005 School of Physics, Peking University, Bachelor degree on Physics and Mathematics Research Interests: Quantum Field Theory, String Theory and Mathematical Physics, with a focus on Scattering Amplitudes and related topics
Citation for the Award (within 30 words)
His contributions in revolutionizing our basic understanding for scattering of particles and strings, and their deep connections to mathematics
Description of the work
Scattering amplitudes have played a central role in Quantum Field Theory (QFT) since its inception, and lie at the heart of our basic understanding of Nature at its smallest scales. In 2003, Witten invented twistor string theory which has revealed rich structures underlying scattering amplitudes in maximally supersymmetric Yang-Mills theory (N=4 SYM) and unexpected connections to strings. Since then there has been enormous progress in computing and understanding S-matrices in gauge theories, gravity and string theory. As one of the leading experts in this fast-growing research field, Prof. Song He has made lots of outstanding contributions in developing new ways for computing scattering amplitudes and unravelling their deep mathematical structures, and he has made several discoveries that have been widely characterized as breakthroughs. These advances have not only enabled more precise predictions of Standard Model at high-energy experiments such as the LHC, but also shed new lights into fundamental questions about QFT, quantum gravity and even mathematics.

Among others, Prof. He has been known for seminal works on scattering amplitudes and Wilson loops in planar N=4 SYM, which is the first four-dimensional QFT that could be solved exactly. In a groundbreaking work with Caron-Huot [1], he has discovered quantum corrections for the infinite-dimensional Yangian symmetry of the theory at any value of the coupling, which has provided a very powerful method for all-loop computations in planar N=4 SYM and has inspired new progress in computing amplitudes and Feynman integrals.

In a series of works with his collaborators starting [2], Prof. He has invented a new formulation for S-matrices known as the Cachazo-He-Yuan (CHY) formula , which has revolutionized our basic understanding of scattering of massless particles such as gluons, gravitons and pions. The CHY formulation has extended Witten's twistor strings to arbitrary space-time dimension and to a wide range of theories including Yang-Mills theory, Einstein gravity and many more; it has revealed hidden simplicity and structures which are completely obscured in the textbook formulation using Lagrangian and Feynman diagrams. These groundbreaking works have opened up a new research direction, and Prof. He has further developed the new formulation and discovered new structures of amplitudes and even new relations between them. For example, he has made important contributions in studying the so-called double copy relations between gauge theories and gravity by extending the field-theory limit of Kawai-Lewellen-Tye (KLT) relations, which express gravity amplitudes as the "square" of Yang-Mills ones, to quantum level.

With Arkani-Hamed and other collaborators, Prof. He has discovered a new geometric picture for scattering amplitudes in QFT even as simple as cubic scalar theory[3]. By realizing the associahedron polytope in kinematic space (known as "ABHY associahedra"), important principles of relativity and quantum mechanics, such as locality and unitarity, emerge naturally for the amplitudes. This has beautifully generalized the famous "amplituhedron" of N=4 SYM to cubic scalar theory in arbitrary dimensions and revealed deep connections with other important ideas such as color-kinematics duality and double copy, CHY and string theory. More recently, they have found such remarkable geometric structures for string amplitudes and their vast generalizations, which are intimately connected with mathematics such as positive geometries and cluster algebras.

In summary, Prof. He has played a leading role in recent breakthroughs on our understanding for scattering of particles and strings, and deep connections with mathematics. Prof. Song He deserves the Nishina Asia Award.

Key references (up to 3 key publications*)

[1] Simon Caron-Huot; **Song He**; Jumpstarting the All-Loop S-Matrix of Planar N=4 Super Yang-Mills, *Journal of High Energy Physics*, 2012, 1207(174).https://arxiv.org/abs/1112.1060

[2] Freddy Cachazo; **Song He**; Ellis Ye Yuan; Scattering of Massless Particles in Arbitrary Dimensions, *Phys.Rev.Lett.*, 2014, 113(17): 171601 https://arxiv.org/abs/1307.2199

[3] Nima Arkani-Hamed; Yuntao Bai; **Song He**; Gongwang Yan; Scattering Forms and Positive Geometry of Kinematics, Color and the Worldsheet, *Journal of High Energy Physics*, 2018, 1805(096). https://arxiv.org/abs/1711.09102

*) Copy of one most significant publication should be attached.

Nominator (name, affiliation, email, telephone and relation to the candidate)

Prof. Ronggen Cai, Director, Institute of Theoretical Physics, Chinese Academy of Sciences
Email:cairg@itp.ac.cn, Tel: 010-62582505
Relation to the candidate: colleague

Signature Rong-Gen Cai Date March 11, 2021

Scattering of Massless Particles in Arbitrary Dimensions

Freddy Cachazo,[1,*] Song He,[1,2,†] and Ellis Ye Yuan[1,3,‡]
[1]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*
[2]*School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540, USA*
[3]*Department of Physics & Astronomy, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*
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We present a compact formula for the complete tree-level S-matrix of pure Yang-Mills and gravity theories in arbitrary spacetime dimensions. The new formula for the scattering of n particles is given by an integral over the positions of n points on a sphere restricted to satisfy a dimension-independent set of equations. The integrand is constructed using the reduced Pfaffian of a $2n\times 2n$ matrix, Ψ, that depends on momenta and polarization vectors. In its simplest form, the gravity integrand is a reduced determinant which is the square of the Pfaffian in the Yang-Mills integrand. Gauge invariance is completely manifest as it follows from a simple property of the Pfaffian.

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Introduction.—In a recent work [1], we pointed out the existence of equations connecting the space of kinematic invariants of n massless particles in any dimension and that of the positions of n points on a sphere. The equations are given by

$$\sum_{b\neq a}\frac{s_{ab}}{\sigma_a-\sigma_b}=0 \quad \text{for } a\in\{1,2,\ldots,n\}, \tag{1}$$

where $s_{ab}=(k_a+k_b)^2=2k_a\cdot k_b$, and σ_c is the position of the cth puncture. Motivated by some remarkable properties of these equations, namely, their connection to general kinematic invariants and Kawai-Lewellen-Tye (KLT) orthogonality [2], it was proposed that they are the backbone of the tree-level S-matrix of massless particles in any dimension and were called the scattering equations.

The scattering equations are invariant under $\mathrm{SL}(2,\mathbb{C})$ transformations when external vectors satisfy momentum conservation. They already made an appearance in previous literature in [3–6], and, in particular, in Ref. [7] by one of the authors in studying fundamental Bern-Carrasco-Johansson (BCJ) relations [8] in four dimensions [9]. The validity of the BCJ relations for gauge-theoretical amplitudes in any dimension [8,10,11] also provides an important piece of evidence for the universal relevance of the scattering equations.

In Ref. [1], we also proposed the existence of formulas for the complete S-matrix of Yang-Mills and gravity theories in any dimension. In this Letter, we provide the explicit construction of such formulas.

Preliminaries.—The first step towards the construction of formulas in any spacetime dimension is that of the measure. Given that only $n-3$ of the n scattering equations are linearly independent, one has to find a way of imposing their support in a permutation-invariant manner. This is achieved by noticing that

$$\prod\nolimits_a{}'\delta\left(\sum_{b\neq a}\frac{s_{ab}}{\sigma_{ab}}\right):=\sigma_{ij}\sigma_{jk}\sigma_{ki}\prod_{a\neq i,j,k}\delta\left(\sum_{b\neq a}\frac{s_{ab}}{\sigma_{ab}}\right) \tag{2}$$

is independent of the choice $\{i,j,k\}$ and, hence, permutation invariant. Here and in the rest of this Letter, $\sigma_{ab}:=\sigma_a-\sigma_b$.

Let us denote the n-gluon partial amplitude with the canonical ordering $1,2,\ldots,n$ as A_n and the n-graviton amplitude as M_n. It is now natural to propose the following formulations of their S matrices:

$$A_n=\int\frac{d^n\sigma}{\mathrm{volSL}(2,\mathbb{C})}\prod\nolimits_a{}'\delta\left(\sum_{b\neq a}\frac{s_{ab}}{\sigma_{ab}}\right)\frac{E_n(\{k,\epsilon,\sigma\})}{\sigma_{12}\cdots\sigma_{n1}}, \tag{3}$$

$$M_n=\int\frac{d^n\sigma}{\mathrm{volSL}(2,\mathbb{C})}\prod\nolimits_a{}'\delta\left(\sum_{b\neq a}\frac{s_{ab}}{\sigma_{ab}}\right)E_n^2(\{k,\epsilon,\sigma\}), \tag{4}$$

where $E_n(\{k,\epsilon,\sigma\})$ is a permutation-invariant function of momenta k_a^μ, polarization vectors ϵ_a^μ, and σ_a. Note that the $\mathrm{SL}(2,\mathbb{C})$ invariance of the integrand constrains E_n: under an $\mathrm{SL}(2,\mathbb{C})$ transformation, $\sigma_a\to(A\sigma_a+B)/(C\sigma_a+D)$, E must transform as

$$E_n(\{k,\epsilon,\sigma\})\to E_n(\{k,\epsilon,\sigma\})\prod_{a=1}^n(C\sigma_a+D)^2. \tag{5}$$

It is also natural to expect that E_n should be gauge invariant for each solution of the scattering equations.

At this point, it is worth to spell out how the measure is computed in practice, which uncovers a beautiful relation to a matrix found previously in the literature and, hence, shows its permutation invariance. Consider the object

$$\int\frac{d^n\sigma}{\mathrm{volSL}(2,\mathbb{C})}\prod\nolimits_a{}'\delta\left(\sum_{b\neq a}\frac{s_{ab}}{\sigma_{ab}}\right)\mathcal{I}, \tag{6}$$

where $\mathcal{I}$ represents either the integrand of Yang-Mills or that of gravity. Using a Fadeev-Popov procedure to gauge fix the $\mathrm{SL}(2,\mathbb{C})$ redundancy and, hence, fix the value of, say, σ_p, σ_q, σ_r, one finds that Eq. (6) becomes

$$\int \prod_{c\neq p,q,r} d\sigma_c(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki}) \prod_{a\neq i,j,k} \delta\left(\sum_{b\neq a}\frac{s_{ab}}{\sigma_{ab}}\right)\mathcal{I}. \quad (7)$$

The delta functions completely localize all integrals. As proven in Ref. [1], the scattering equations have $(n-3)!$ solutions, and the answer is obtained by evaluating a Jacobian and the integrand on them. The Jacobian can be computed by starting with an $n\times n$ matrix Φ defined by

$$\Phi_{ab} = \begin{cases} \frac{s_{ab}}{(\sigma_a-\sigma_b)^2}, & a\neq b \\ -\sum\limits_{c\neq a}\frac{s_{ac}}{(\sigma_a-\sigma_c)^2}, & a=b. \end{cases} \quad (8)$$

The fact that the delta functions exclude $\{i,j,k\}$ means that we have to delete those rows from Φ while having fixed the values of σ_p, σ_q, σ_r means that we have to delete columns $\{p,q,r\}$. Let us denote the corresponding minor by $|\Phi|^{ijk}_{pqr}$. This minor is the Jacobian we are after. The answer is then

$$\sum_{\{\sigma\}\in\,\text{solutions}} \frac{(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki})}{|\Phi|^{ijk}_{pqr}}\mathcal{I}. \quad (9)$$

Precisely the combination that appears in this equation is what was called $\det'\Phi$ by Cachazo and Geyer in Ref. [12] (inspired by a remarkable formula for maximally-helicity-violating gravity amplitudes found by Hodges in Ref. [13]) and which is known to be completely permutation invariant, i.e., independent of the choices made in selecting $\{i,j,k\}$ and $\{p,q,r\}$. More explicitly,

$$\det{}'\Phi := \frac{|\Phi|^{ijk}_{pqr}}{(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki})}. \quad (10)$$

Explicit form of $E_n(\{k,\epsilon,\sigma\})$.—In order to present the explicit form of $E_n(\{k,\epsilon,\sigma\})$, we first define the following $2n\times 2n$ antisymmetric matrix:

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad (11)$$

where A, B, and C are $n\times n$ matrices. The first two matrices have components

$$A_{ab} = \begin{cases} \frac{s_{ab}}{\sigma_a-\sigma_b}, & a\neq b \\ 0, & a=b, \end{cases} \qquad B_{ab} = \begin{cases} \frac{2\epsilon_a\cdot\epsilon_b}{\sigma_a-\sigma_b}, & a\neq b \\ 0, & a=b, \end{cases} \quad (12)$$

while the third is given by

$$C_{ab} = \begin{cases} \frac{2\epsilon_a\cdot k_b}{\sigma_a-\sigma_b}, & a\neq b \\ -\sum\limits_{c\neq a}\frac{2\epsilon_a\cdot k_c}{\sigma_a-\sigma_c}, & a=b. \end{cases} \quad (13)$$

The first important observation is that while the Pfaffian of Ψ is zero, removing rows i,j, and columns i,j with $1\leq i<j\leq n$ gives rise to a new matrix Ψ^{ij}_{ij} with nonzero Pfaffian and such that

$$\mathrm{Pf}'\Psi := \frac{(-1)^{i+j}}{(\sigma_i-\sigma_j)}\mathrm{Pf}(\Psi^{ij}_{ij}) \quad (14)$$

is independent of the choice of i and j. We call $\mathrm{Pf}'\Psi$ the reduced Pfaffian of Ψ.

The Pfaffian of Ψ vanishes because its first n rows (or columns) are linearly dependent: actually, the $n\times 2n$ matrix $(A,-C^T)$ has two null vectors, $(1,\ldots,1)$ and $(\sigma_1,\ldots,\sigma_n)$, thus, $(\mathrm{Pf}\Psi)^2=\det\Psi=0$. Now we turn to the proof that the reduced Pfaffian is invariant under permutations of particle labels. First note that simultaneous interchanges of two columns and rows change the sign of the Pfaffian. When exchanging two particle labels a,b, which are different from i,j, we must exchange rows and columns a,b and also exchange $a+n,b+n$; when exchanging particle labels i,j, we only exchange $i+n$, $j+n$ in Ψ^{ij}_{ij}, and the additional minus sign cancels with the minus sign from the prefactor in the definition of $\mathrm{Pf}'\Psi$. Hence, in both cases the reduced Pfaffian is invariant. Therefore, to prove permutation invariance, it suffices to prove that the reduced Pfaffian obtained from removing columns and rows 1, 2 and that from removing 1, 3 are identical.

We multiply the first row and column of Ψ^{12}_{12} by σ_{13}, and the first row and column of Ψ^{13}_{13} by σ_{12}, and obtain two matrices we call Ψ'^{12}_{12} and Ψ'^{13}_{13}. Next, we take a multiple of the $(i-2)$th row and column of Ψ'^{12}_{12} by σ_{1i} and add all the multiples to the first row and column, respectively, for $i=4,\ldots,n$; in this way, we obtain a new matrix Ψ''^{12}_{12}, and similarly, we have Ψ''^{13}_{13}, whose Pfaffians are related to the original ones by $\mathrm{Pf}\Psi''^{12}_{12}=\sigma_{13}\mathrm{Pf}\Psi^{12}_{12}$, $\mathrm{Pf}\Psi''^{13}_{13}=\sigma_{12}\mathrm{Pf}\Psi^{13}_{13}$. By the scattering equations, it is straightforward to show that the first row and column of Ψ''^{12}_{12} only differ from the first row and column of Ψ''^{13}_{13} by a minus sign; note that other columns and rows of the two new matrices are identical, thus, $\mathrm{Pf}\Psi^{12}_{12}/\sigma_{12}=-(\mathrm{Pf}\Psi^{13}_{13})/(\sigma_{13})$. We conclude that the reduced Pfaffian $\mathrm{Pf}'\Psi$ is permutation invariant with respect to the particle labels.

Now we are ready to write down the proposal

$$E_n(\{k,\epsilon,\sigma\}) = \mathrm{Pf}'\Psi(\{k,\epsilon,\sigma\}). \quad (15)$$

Combining this proposal for E_n with the general formula (3) and (4) gives the main results of this Letter: a formula for the tree-level S-matrix of Yang-Mills in any dimension

$$A_n = \frac{1}{\mathrm{vol\,SL}(2,\mathbb{C})}\int \frac{d^n\sigma}{\sigma_{12}\cdots\sigma_{n1}}\prod_a{}'\delta\left(\sum_{b\neq a}\frac{s_{ab}}{\sigma_{ab}}\right)\mathrm{Pf}'\Psi. \quad (16)$$

And, using the KLT construction in the form discussed in Refs. [1,12] and the KLT orthogonality proven in Ref. [1], a formula for the tree-level S-matrix of gravity

$$M_n = \frac{1}{\mathrm{vol\,SL}(2,\mathbb{C})}\int d^n\sigma\prod_a{}'\delta\left(\sum_{b\neq a}\frac{s_{ab}}{\sigma_{ab}}\right)\mathrm{Pf}'\Psi\mathrm{Pf}'\tilde{\Psi}. \quad (17)$$

Here, $\tilde{\Psi}$ is taken to mean $\Psi(k,\tilde{\epsilon},\sigma)$ and where $\tilde{\epsilon}_a$ represents the same physical polarization as ϵ_a. In its simplest form, one can choose $\tilde{\epsilon}_a = \epsilon_a$ and obtain

$$M_n = \frac{1}{\mathrm{vol\,SL}(2,\mathbb{C})}\int d^n\sigma\prod_a{}'\delta\left(\sum_{b\neq a}\frac{s_{ab}}{\sigma_{ab}}\right)\det{}'\Psi, \quad (18)$$

where $\det'\Psi$ is defined as $\det\Psi^{ij}_{ij}/\sigma_{ij}^2$.

Finally, it is worth to also write both formulas in a form where all the integrals have been performed using Eqs. (9) and (10)

$$A_n = \sum_{\{\sigma\}\in\mathrm{solutions}}\frac{1}{\sigma_{12}\cdots\sigma_{n1}}\frac{\mathrm{Pf}'\Psi(\{k,\epsilon,\sigma\})}{\det'\Phi} \quad (19)$$

and

$$M_n = \sum_{\{\sigma\}\in\mathrm{solutions}}\frac{\det'\Psi(\{k,\epsilon,\sigma\})}{\det'\Phi}. \quad (20)$$

Properties and checks.—Simple properties such as multilinearity in polarization vectors, $\mathrm{SL}(2,\mathbb{C})$ invariance, and its mass dimension are easy to check by using the expansion of the Pfaffian or its recursion relation (analogous to those of the determinant). Gauge invariance, as the statement that the amplitude vanishes when any ϵ_a^μ is replaced by a multiple of k_a^μ, is obvious since two columns of the matrix Ψ and, hence, of Ψ^{ij}_{ij} become multiples of each other under the replacement. More explicitly, assume that ϵ_i^μ is replaced by k_i^μ, then it can be easily seen that columns i and $i+n$ of Ψ become identical after realizing that

$$C_{ii} = -\sum_{c\neq i}\frac{2\epsilon_i\cdot k_c}{\sigma_i-\sigma_c} \to -\sum_{c\neq i}\frac{2k_i\cdot k_c}{\sigma_i-\sigma_c} = 0 \quad (21)$$

by the scattering equations. The last property which is manifest is the behavior under soft limits. As discussed in detail in Ref. [1], when we take $k_n \to 0$, $n-1$ of the scattering equations become identical to those of a system with $n-1$ particles. The last equation

$$\sum_{b\neq n}\frac{s_{nb}}{\sigma_n-\sigma_b} = 0 \quad (22)$$

becomes a polynomial for σ_n of degree $n-3$ (due to momentum conservation). In this discussion, we focus on Yang-Mills amplitudes. It is convenient to compute $\mathrm{Pf}'\Psi$ using Ψ^{ij}_{ij} with $i\neq n$, $j\neq n$. The Pfaffian of a $2m\times 2m$ matrix E satisfies a recursion relation of the form $\mathrm{Pf}(E) = \sum_{q=1}^{2m}(-1)^q e_{pq}\mathrm{Pf}(E^{pq}_{pq})$. Using this formula to expand $\mathrm{Pf}\Psi^{ij}_{ij}$ setting $p=n$, one finds that in the soft limit, only one term contributes and gives

$$\mathrm{Pf}\Psi^{ij}_{ij} \to C_{nn}\mathrm{Pf}\Psi^{ijn(2n)}_{ijn(2n)}. \quad (23)$$

Very nicely, $\mathrm{Pf}\Psi^{ijn(2n)}_{ijn(2n)}$ is independent of k_n and ϵ_n and leads to $\mathrm{Pf}'\Psi_{n-1}$, i.e., the reduced Pfaffian for $n-1$ particles. Using the explicit formula (16) in the soft limit, one finds

$$A_n \to \sum_{i=1}^{(n-4)!}\oint_\Gamma d\sigma_n\frac{\sum_{a\neq n}\frac{2\epsilon_n\cdot k_a}{\sigma_{na}}}{\sum_{a\neq n}\frac{2k_n\cdot k_a}{\sigma_{na}}}\frac{\sigma_{n-1,1}}{\sigma_{n-1,n}\sigma_{n,1}}\mathcal{I}^{(i)}_{n-1}, \quad (24)$$

where $\mathcal{I}^{(i)}_{n-1}$ are the terms in the expansion of Eq. (19) for A_{n-1}, and all σ_a's with $a\in\{1,2,\ldots,n-1\}$ are taken to be evaluated on the ith solution. Also, the contour Γ is defined to encircle the $n-3$ zeroes of the first factor in the denominator [14]. Using the residue theorem, one finds that there is no contribution at infinity, and only two poles have nonvanishing residue. These are at $\sigma_n = \sigma_{n-1}$ and at $\sigma_n = \sigma_1$. The residues are trivial to compute as only one term from the sum in the numerator and one from that in the denominator contribute, giving rise to

$$A_n \to \left(\frac{\epsilon_n\cdot k_{n-1}}{k_n\cdot k_{n-1}} + \frac{\epsilon_n\cdot k_1}{k_n\cdot k_1}\right)A_{n-1}, \quad (25)$$

which is the correct soft behavior [15]. A completely analogous computation gives the correct soft behavior for gravity as well. Factorization of the amplitude on physical poles is a more involved computation and details are provided in Ref. [16].

We have also performed some nontrivial checks, such as the agreement of our formula for gluons with formulas available in the literature for three-, four-, and five-particle scattering in any dimension (see, e.g., Ref. [17] for $n=3,4$ and Ref. [18] for $n=5$). The case with five particles is the most interesting one as the scattering equations in dimensions greater than four do not factor, and the two solutions come from an irreducible quadratic equation. This is the first case that our formula clearly computes the amplitudes in a novel way. We also performed numerical checks that when evaluated in four-dimensional kinematics, our formula reproduces all amplitudes with $n\leq 8$ and in all possible helicity sectors (including the all plus and all but one plus).

Discussion.—We have presented a formula for the complete tree-level S-matrix of gluons and gravitons in any spacetime dimension. While formulas in dimensions less than ten could exploit the presence of supersymmetry in defining an on-shell superspace, such as the formula of Witten and of Roiban, Spradlin, and Volovich does in four dimensions [19], our formula necessarily depends on polarization vectors as it is also valid in dimensions where supersymmetry does not exist. Any formula which contains polarization vectors must satisfy the constraint that it vanishes when any polarization vector is replaced by a multiple of its momentum vector. What we have found in this work is that there exists a very compact formula in which gauge invariance is actually a simple property of its intrinsic structure, and, indeed, it was the main clue for its derivation.

As discussed in Ref. [20] and more recently in Refs. [1,21], there are compact formulas for string amplitudes in terms of Yang-Mills or gravity amplitudes, and our proposal here also provides a simple representation of string amplitudes in terms of polarization vectors. In relation to string amplitudes, it is important to mention an intriguing connection to their high energy scattering limit. In the work of Gross and Mende [4], the scattering equations appear as the conditions imposed by the saddle point evaluation of the string amplitude. It is tempting to suggest that this is more than a coincidence.

Finally, also worth mentioning is that all $(n-3)!$ solutions of the scattering equations give rise to gauge-invariant contributions. Moreover, under factorization limits, each term in Eqs. (19) and (20) either develops a pole and "factors," or it remains finite. This is reminiscent of the behavior of partial amplitudes in Yang-Mills theory where a decomposition of the full amplitude is made in parts that do not exhibit all factorization channels. Adding the fact that in dimensions greater than four the $(n-3)!$ do not split into sectors, it is natural to suggest that each solution is a "partial amplitude." In Yang-Mills theory, this decomposition is in addition to the usual color ordering one, while in gravity, it is all there is. It would be fascinating to fully uncover the physical meaning of this new decomposition.

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Note added.—Recently, more detailed discussions of our prescription and its application in scalar theories were summarized in Ref. [22], the validity of our formula was proven by Ref. [23] using Britto-Cachazo-Feng-Witten recursion relations [24,25], and ambitwistor string models were constructed in Ref. [26] that produce our formula.

*fcachazo@perimeterinstitute.ca
†she@perimeterinstitute.ca
‡yyuan@perimeterinstitute.ca

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